Optimized and secure pairing-friendly elliptic curve suitable for one layer proof composition

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Overview

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   - Zero-knowledge proof
   - ZK-SNARK
2. Proof composition
   - Notations
   - Techniques
3. Our work
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   - Implementation
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Zero-knowledge proof

What is a zero-knowledge proof?

"I have a *sound*, *complete* and *zero-knowledge* proof that a statement is true".

Sound

If the statement is false, no cheating prover can convince an honest verifier that it is true, except with some small probability.

Complete

If the statement is true, an honest verifier will be convinced of this fact by an honest prover.

Zero-knowledge

If the statement is true, no verifier learns anything other than the fact that the statement is true.

Zero-knowledge proof

ZK-SNARK: Zero-Knowledge Succinct Non-interactive ARgument of Knowledge

"I have a *computationally sound*, *complete*, *zero-knowledge*, *succinct*, *non-interactive* proof that a statement is true and that I know a related secret".

Succinct

An honestly-generated proof is very "short" and "easy" to verify.

Non-interactive

No interaction is necessary between the prover and the verifier in order to respectively generate the proof and verify it.

ARgument of Knowledge

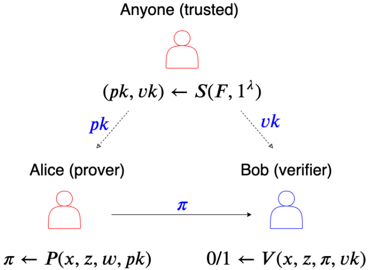
An honest verifier is convinced that a comptutationally bounded prover knows a secret information.

Zero-knowledge proof

Preprocessing ZK-SNARK of NP language

Let F be a public NP program, x and z be public inputs, and w be a private input such that $z := F(x, w)$.
A ZK-SNARK consists of algorithms S, P, V s.t. for a security parameter λ:

Trapdoored Setup:	(pk, vk)	$\leftarrow$	$S(F, \tau, 1^\lambda)$
Prove:	π	$\leftarrow$	$P(x, z, w, pk)$
Verify:	$0/1$	$\leftarrow$	$V(x, z, \pi, vk)$



ZK-SNARK

Succinctness: An honestly-generated proof is very "short" and "easy" to verify.

Definition [BCTV14b]

A succinct proof π has size $O_\lambda(1)$ and can be verified in time $O_\lambda(|F| + |x| + |z|)$, where $O_\lambda(.)$ is some polynomial in the security parameter λ.

- $E\colon y^2 = x^3 + ax + b$ elliptic curve defined over $\mathbb{F}_q$, q a prime power.
- r prime divisor of $\#E(\mathbb{F}_q) = q + 1 - t$, t Frobenius trace.
- $-D$ CM discriminant, $4q = t^2 + Dy^2$ for some integer y.
- d degree of twist.
- k embedding degree, smallest integer $k \in \mathbb{N}^*$ s.t. $r \mid q^k - 1$.
- $\mathbb{G}_1 \subset E(\mathbb{F}_q)$ and $\mathbb{G}_2 \subset E(\mathbb{F}_{q^k})$ two groups of order r.
- $\mathbb{G}_T \subset \mathbb{F}_{q^k}^*$ group of r-th roots of unity.
- pairing $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$.

Example: Groth16 [Gro16]

Given an instance $\Phi = (a_0, \ldots, a_\ell) \in \mathbb{F}_r^\ell$ of a public NP program F

- $(pk, vk) \leftarrow S(F, \tau, 1^\lambda)$ where

$$vk = (vk_{\alpha,\beta}, \{vk_{\pi_i}\}_{i=0}^{\ell}, vk_\gamma, vk_\delta) \in \mathbb{G}_T \times \mathbb{G}_1^{\ell+1} \times \mathbb{G}_2 \times \mathbb{G}_2$$

- $\pi \leftarrow P(\Phi, w, pk)$ where

$$\pi = (A, B, C) \in \mathbb{G}_1 \times \mathbb{G}_2 \times \mathbb{G}_1 \qquad (O_\lambda(1))$$

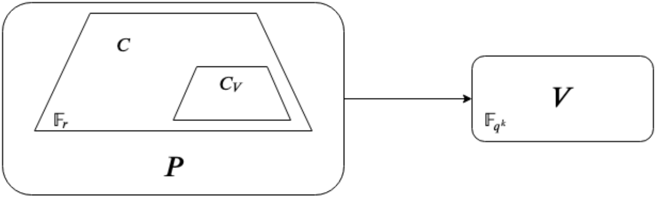
- $0/1 \leftarrow V(\Phi, \pi, vk)$ where V is

$$e(A, B) = vk_{\alpha,\beta} \cdot e(vk_x, vk_\gamma) \cdot e(C, vk_\delta) \qquad (O_\lambda(|\Phi|)) \qquad (1)$$

  and $vk_x = \sum_{i=0}^{\ell} [a_i] vk_{\pi_i}$ depends only on the instance Φ and $vk_{\alpha,\beta} = e(vk_\alpha, vk_\beta)$ can be computed in the trusted setup for $(vk_\alpha, vk_\beta) \in \mathbb{G}_1 \times \mathbb{G}_2$.

Proof composition

A proof of a proof



Since the verification algorithm V (Eq. 1) is a $\mathtt{NP}$ program, generate a new proof that verifies the correctness the old proof.
Remember that, for pairing-based SNARKs, Eq. 1 is in $\mathbb{F}_{q^k}$ and Φ in $\mathbb{F}_r$, where q is the field size of an elliptic curve E and r its prime subgroup order.

- 1st attempt: choose a curve for which $q = r$ (impossible)
- 2nd attempt: simulate $\mathbb{F}_q$ operations via $\mathbb{F}_r$ operations ($\times \log q$ blowup)
- 3rd attempt: use a cycle/chain of pairing-friendly elliptic curves [BCTV14a, BCG+20]

Definition

An m-chain of elliptic curves is a list of distinct curves $E_1/\mathbb{F}_{q_1}, \ldots,$ $E_m/\mathbb{F}_{q_m}$ where $q_1, \ldots, q_m$ are large primes and

$$\#E_1(\mathbb{F}_{q_1}) = q_2, \ldots, \#E_i(\mathbb{F}_{q_i}) = q_{i+1}, \ldots, \#E_{m-1}(\mathbb{F}_{q_{m-1}}) = q_m$$

Definition

An m-cycle of elliptic curves is a list of distinct curves $E_1/\mathbb{F}_{q_1}, \ldots,$ $E_m/\mathbb{F}_{q_m}$ where $q_1, \ldots, q_m$ are large primes and

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$$\#E_m(\mathbb{F}_{q_m}) = q_1$$

Proof composition

cycles and chains of pairing-friendly elliptic curves

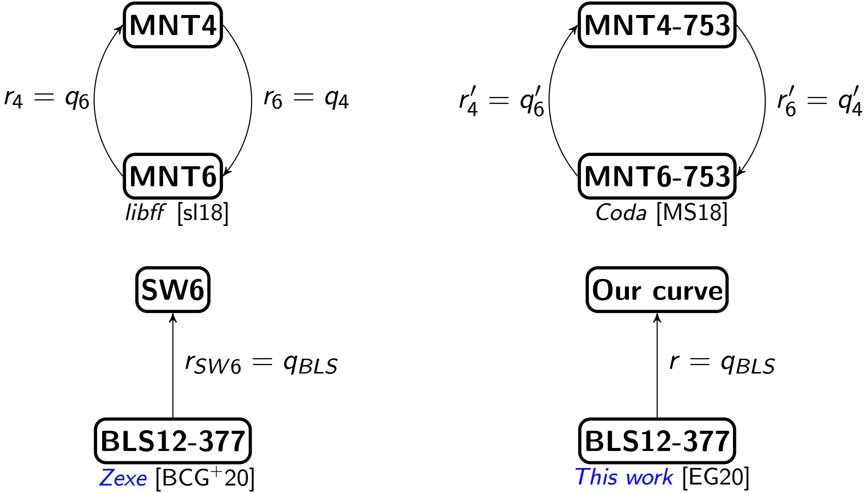

Figure: Examples of pairing-friendly amicable cycles and chains.

Proof composition

cycles and chains of pairing-friendly elliptic curves

$E/\mathbb{F}_q$	q	r	k	d	a, b	λ
MNT4	$q_4 = r_6$ (298b)	$r_4 = q_6$ (298b)	4	2	$a = 2, b = *$	32
MNT6	$q_6 = r_4$ (298b)	$r_6 = q_4$ (298b)	6	2	$a = 11, b = *$	50
MNT4-753	$q'_4 = r'_6$ (753b)	$r'_4 = q'_6$ (753b)	4	2	$a = 2, b = *$	128
MNT6-753	$q'_6 = r'_4$ (753b)	$r'_6 = q'_4$ (753b)	6	2	$a = 11, b = *$	128
BLS12-377	q_{BLS} (377b)	r_{BLS} (253b)	12	6	$a = 0, b = 1$	128
SW6	q_{SW6} (782b)	$r_{SW6} = q_{BLS}$ (377b)	6	2	$a = 5, b = *$	128
This work	q (761b)	$r = q_{BLS}$ (377b)	6	6	$a = 0, b = -1$	128

Table: 2-cycle and 2-chain examples.

Recall that $E/\mathbb{F}_q : y^2 = x^3 + ax + b$ has a subgroup of order r, an embedding degree k, a twist of order d and an approximate security of λ-bit.

- SNARK
  - $E/\mathbb{F}_q$ — BN, BLS12, BW12?, KSS16? ... [FST10]
    - pairing-friendly
    - $r-1$ highly 2-adic
- Recursive SNARK (2-cycle)
  - $E_1/\mathbb{F}_{q_1}$ and $E_2/\mathbb{F}_{q_2}$ — MNT4/MNT6 [FST10, Sec.5], ? [CCW19]
    - both pairing-friendly
    - $r_2 = q_1$ and $r_1 = q_2$
    - $r_{\{1,2\}} - 1$ highly 2-adic
    - $q_{\{1,2\}} - 1$ highly 2-adic
- Recursive SNARK (2-chain)
  - $E_1/\mathbb{F}_{q_1}$ — BLS12 ($seed \equiv 1 \mod 3.2^{adicity}$) [BCG+20], ?
    - pairing-friendly
    - $r_1 - 1$ highly 2-adic
    - $q_1 - 1$ highly 2-adic
  - $E_2/\mathbb{F}_{q_2}$ — Cocks–Pinch algorithm
    - pairing-friendly
    - $r_2 = q_1$

Our work

Snarky curve $E_2/\mathbb{F}_{q_2}$

- q is a prime or a prime power
- t is relatively prime to q
- ~~r is prime~~
- ~~r divides $q+1-t$~~
- ~~r divides q^k-1 (smallest $k \in \mathbb{N}^*$)~~
- $4q-t^2 = Dy^2$ (for $D < 10^{12}$) and some integer y

r is a **fixed** chosen prime that divides $q+1-t$ and q^k-1 (smallest $k \in \mathbb{N}^*$)

Algorithm 1: Cocks–Pinch method

1 Fix k and D and choose a prime r s.t. $k|r-1$ and $(\frac{-D}{r})=1$;
2 Compute $t = 1 + x^{(r-1)/k}$ for x a generator of $(\mathbb{Z}/r\mathbb{Z})^\times$;
3 Compute $y = (t-2)/\sqrt{-D} \bmod r$;
4 Lift t and y in $\mathbb{Z}$;
5 Compute $q = (t^2 + Dy^2)/4$ (in $\mathbb{Q}$);
6 back to 1 if q is not a prime integer;

Our work

Limitations and improvements over CP

- $\rho = \log_2 q / \log_2 r \approx 2$ (because $q = f(t^2, y^2)$ and $t, y \xleftarrow{\$} \bmod r$).
- The curve parameters (q, r, t) are not expressed as polynomials.

Algorithm 2: Brezing–Weng method

1 Fix k and D and choose an irreducible polynomial $r(x) \in \mathbb{Z}[x]$ with positive leading coefficient [1] s.t. $\sqrt{-D}$ and the primitive k-th root of unity ζ_k are in $K = \mathbb{Q}[x]/r(x)$;

2 Choose $t(x) \in \mathbb{Q}[x]$ be a polynomial representing $\zeta_k + 1$ in K;

3 Set $y(x) \in \mathbb{Q}[x]$ be a polynomial mapping to $(\zeta_k - 1)/\sqrt{-D}$ in K;

4 Compute $q(x) = (t^2(x) + Dy^2(x))/4$ in $\mathbb{Q}[x]$;

- $\rho = 2 \max(\deg t(x), \deg y(x)) / \deg r(x) < 2$
- $r(x), q(x), t(x)$ but $\exists x_0 \in \mathbb{Z}^*, r(x_0) = r_{fixed}$ and $q(x_0)$ is prime ?

[1] conditions to satisfy Bunyakovsky conjecture which states that such a polynomial produces infinitely many primes for infinitely many integers.

- $\mathbb{G}_2 \subset E(\mathbb{F}_{q^k}) \cong E'[r](\mathbb{F}_{q^{k/d}})$ for a twist E' of degree d.
- When $-D = -3$, there exists a twist E' of degree $d = 6$.
- Associated with a choice of $\xi \in \mathbb{F}_{q^{k/6}}$ s.t. $x^6 - \xi \in \mathbb{F}_{q^{k/6}}[x]$ is irreducible, the equation of E' can be either
  - $y^2 = x^3 + b/\xi$ and we call it a D-twist or
  - $y^2 = x^3 + b.\xi$ and we call it a M-twist.
- For the D-type, $E' \to E : (x, y) \mapsto (\xi^{1/3}x, \xi^{1/2}y)$,
- For the M-type $E' \to E : (x, y) \mapsto (\xi^{2/3}x/\xi, \xi^{1/2}y/\xi)$

Our work

Suggested construction: combines CP and BW

1. Cocks–Pinch method
   - $k = 6$ and $-D = -3 \implies$ 128-bit security, $\mathbb{G}_2$ coordinates in $\mathbb{F}_q$, GLV multiplication over $\mathbb{G}_1$ and $\mathbb{G}_2$
   - restrict search to size$(q) \leq 768$ bits $\implies$ smallest machine-word size
2. Brezing–Weng method
   - choose $r(x) = q_{BLS12-377}(x)$
   - $q(x) = (t^2(x) + 3y^2(x))/4$ is reducible $\implies$ $q(x_0)$ cannot be prime
   - lift $t = r \times h_t + t(x_0)$ and $y = r \times h_y + y(x_0)$ [FK19, GMT20]

We found the following curve $E : y^2 = x^3 - 1$ over $\mathbb{F}_q$ of 761-bit. The parameters are expressed in polynomial forms and evaluated at the seed $x_0 = \mathtt{0x8508c00000000}$. For pairing computation we use the M-twist curve $E' : y^2 = x^3 + 4$ over $\mathbb{F}_q$ to represent $\mathbb{G}_2$ coordinates.

Our curve, $k = 6$, $D = 3$, $r = q_{BLS12-377}$
$r(x) = (x^6 - 2x^5 + 2x^3 + x + 1)/3 = q_{BLS12-377}(x)$
$t(x) = x^5 - 3x^4 + 3x^3 - x + 3 + h_t r(x)$
$y(x) = (x^5 - 3x^4 + 3x^3 - x + 3)/3 + h_y r(x)$
$q(x) = (t^2 + 3y^2)/4$
$q_{h_t=13,h_y=9}(x) = (103x^{12} - 379x^{11} + 250x^{10} + 691x^9 - 911x^8$
$-79x^7 + 623x^6 - 640x^5 + 274x^4 + 763x^3 + 73x^2 + 254x + 229)/9$

- The curve is over 761-bit instead of 782-bit, we save one machine-word of 64 bits.
- The curve has an embedding degree $k = 6$ and a twist of order $d = 6$, allowing $\mathbb{G}_2$ coordinates to be in $\mathbb{F}_q$ (factor 6 compression).
- The curve parameters have polynomial expressions, allowing fast implementation.
- The curve has a very efficient optimal ate pairing.
- The curve has CM discriminant $-D = -3$, allowing fast GLV multiplication on both $\mathbb{G}_1$ and $\mathbb{G}_2$.
- The curve has fast subgroup checks and fast cofactor multiplication on $\mathbb{G}_1$ and $\mathbb{G}_2$ via endomorphisms.
- The curve has fast and secure hash-to-curve methods for both $\mathbb{G}_1$ and $\mathbb{G}_2$.

Our work

Cost estimation of a pairing

$e(P,Q) = f_{t-1,Q}(P)^{(q^6-1)/r}$ $(t-1)$ of 388 bits, $Q \in \mathbb{F}_{q^3}$

$e(P,Q) = (f_{x_0+1,Q}(P) f^q_{x_0^3-x_0^2-x_0,Q}(P))^{(q^6-1)/r}$ x_0 of 64 bits, $Q \in \mathbb{F}_q$

$$(q^6-1)/r = \underbrace{(q^3-1)(q+1)}_{\text{easy part}}\underbrace{(q^2-q+1)/r}_{\text{hard part}} = \begin{cases} \text{easy part} \times (w_0 + q w_1) \\ \text{easy part} \times f(x_0, q^i) \end{cases}$$

Curve	Prime	Pairing	Miller loop	Exponentiation	Total
BLS12	377-bit	ate	6705 m_{384}	7063 m_{384}	13768 m_{384}
SW6	782-bit	ate	47298 m_{832}	10521 m_{832}	57819 m_{832}
This	761-bit	opt. ate	7911 m_{768}	5081 m_{768}	12992 m_{768}

m_b base field multiplication, b bitsize in Montgomery domain on a 64-bit platform

x4.5 less operations in a smaller field by one machine-word

Our work

C++ implementation timings

Implemented in `libff` library [sl18] (with GMP 6.1.2_2) and tested on a 2.2 GHz Intel Core i7 x86_64 processor with 16 Go 2400 MHz DDR4 memory running macOS Mojave 10.14.6. C++ compiler is clang 10.0.1. Profiling routines use `clock_gettime` and `readproc` calls.

url: `https://github.com/EYBlockchain/zk-swap-libff/tree/ey/libff/algebra/curves/bw6_761`

Curve	Pairing	Miller loop	Exponentiation	Total	Eq. 1
BLS12	ate	0.0025s	0.0049s	0.0074s	0.0149s
SW6	ate (proj.)	0.0388s	0.0110s	0.0499s	0.1274s
SW6	ate (aff.)	0.0249s	0.0110s	0.0361s	0.0875s
This	opt. ate	0.0053s	0.0044s	0.0097s	0.0203s

x5 faster to compute a pairing (in projective coordinates)
x6.27 faster to verify a Groth16 proof (in projective coordinates)
x3.7 faster to compute a pairing (in affine coordinates)
x4.22 faster to verify a Groth16 proof (in affine coordinates)

- Zexe: user-defined assets, decentralized exchanges and policy-enforcing stablecoins
- Celo: batched verification of BLS signatures
- Filecoin: circuit splitting
- Baseline protocol (EY, Consensys) [ECM20]: batching zkSNARK proofs

paper: `ia.cr/2020/351`

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