Pairings in R1CS

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zkSummit 7 - April 21, 2022

Overview

Overview

SNARKs examples: Groth16 and PLONK

- m = number of wires
- n = number of multiplications gates
- a = number of additions gates
- ℓ = number of public inputs
- $\mathtt{M}_{\mathbb{G}}$ = multiplication in $\mathbb{G}$
- P=pairing

	Setup	Prove	Verify
Groth16 [Gro16]	$3n$ $\mathtt{M}_{\mathbb{G}_1}$ m $\mathtt{M}_{\mathbb{G}_2}$	$(3n+m-\ell)$ $\mathtt{M}_{\mathbb{G}_1}$ n $\mathtt{M}_{\mathbb{G}_2}$ 7 FFT	3 P ℓ $\mathtt{M}_{\mathbb{G}_1}$
PLONK (KZG) [GWC]	$d_{\geq n+a}$ $\mathtt{M}_{\mathbb{G}_1}$ 1 $\mathtt{M}_{\mathbb{G}_2}$ 8 FFT	$9(n+a)$ $\mathtt{M}_{\mathbb{G}_1}$ 8 FFT	2 P 18 $\mathtt{M}_{\mathbb{G}_1}$

Bilinear pairings

- $E\colon y^2 = x^3 + ax + b$ elliptic curve defined over $\mathbb{F}_q$, q a prime power.
- r prime divisor of $\#E(\mathbb{F}_q) = q + 1 - t$, t Frobenius trace.
- k embedding degree, smallest integer $k \in \mathbb{N}^*$ s.t. $r \mid q^k - 1$.
- a bilinear pairing

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$$

  - $\mathbb{G}_1 \subset E(\mathbb{F}_q)$ a group of order r
  - $\mathbb{G}_2 \subset E(\mathbb{F}_{q^k})$ a group of order r
  - $\mathbb{G}_T \subset \mathbb{F}_{q^k}^*$ group of r-th roots of unity

Overview

- **Proof aggregation or**
- **Private computation (ZEXE)**
  e.g. G16 proof $\pi = (A, B, C) \in \mathbb{G}_1 \times \mathbb{G}_2 \times \mathbb{G}_1$
  and $vk = (vk_1, vk_2, vk_3, vk_4) \in \mathbb{G}_T \times \mathbb{G}_1^{\ell+1} \times \mathbb{G}_2 \times \mathbb{G}_2$

$$\mathtt{V}: \quad e(A, B) \stackrel{?}{=} vk_1 \cdot e(vk_2', vk_3) \cdot e(C, vk_4) \qquad (O_\lambda(\ell)) \tag{1}$$

and $vk_2' = \sum_{i=0}^{\ell} [x_i] vk_2$.

- **BLS signatures**

$$\mathtt{V}: \quad e(\sigma, \mathbb{G}_2) \stackrel{?}{=} e(H(m), Q_{pk}) \qquad (2)$$

where $\sigma \in \mathbb{G}_1$ is the signature, $H(m)$ the message hashed into $\mathbb{G}_1$ and Q_{pk} the public key of the sender.

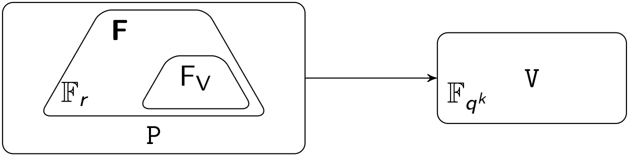
- **Proof of KZG verification (zkEVM)**
  Proof of $P(z) = y$ ($P \in \mathbb{F}_r[X]$)

$$\texttt{V}: \quad e(\pi, vk - [z]\mathbb{G}_2) \stackrel{?}{=} e(C - [y]\mathbb{G}_1, \mathbb{G}_2) \tag{3}$$

where $C \in \mathbb{G}_1$ is the commitment and $vk \in \mathbb{G}_1$ the verification key.

Pairings in SNARKs

An arithmetic mismatch



- **F** any program is expressed in $\mathbb{F}_r$
- P proving is performed over $\mathbb{F}_r[X]$ and $\mathbb{G}_1$ (and $\mathbb{G}_2$)
- V verification (eq. 1, 2 and 3) is done in $\mathbb{F}^*_{q^k}$
- F_V programs of V are natively expressed in $\mathbb{F}^*_{q^k}$ not $\mathbb{F}_r$

Pairings in SNARKs

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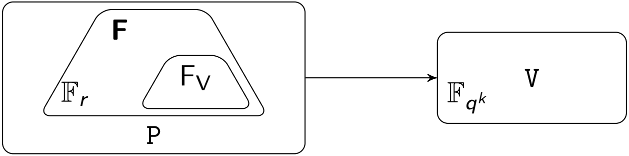

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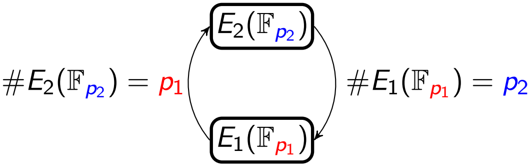
F_V programs of V are natively expressed in $\mathbb{F}^*_{q^k}$ not $\mathbb{F}_r$

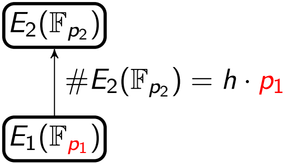
- 1st attempt: choose a curve for which $q = r$ (impossible)
- 2nd attempt: simulate $\mathbb{F}_q$ operations via $\mathbb{F}_r$ operations ($\times \log q$ blowup)
- 3rd attempt: use a cycle/chain of pairing-friendly elliptic curves [CFH+15, BCTV14, BCG+20]

Pairings in SNARKs: solutions

A *cycle* of elliptic curves:

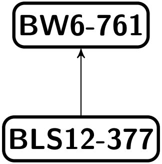


A *2-chain* of elliptic curves:

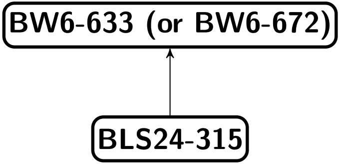


Eurocrypt 2022 [EG22]

Groth16



KZG



Overview

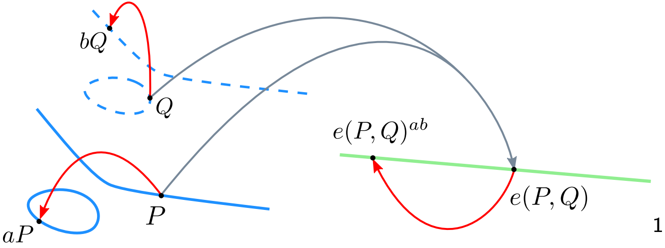
Pairings out-circuit

A non-degenerate bilinear pairing $e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_T$

non-degenerate: $\forall P \in \mathbb{G}_1,\ P \neq \mathcal{O},\ \exists Q \in \mathbb{G}_2, e(P,Q) \neq 1_{\mathbb{G}_T}$

$\forall Q \in \mathbb{G}_2,\ Q \neq \mathcal{O},\ \exists P \in \mathbb{G}_1, e(P,Q) \neq 1_{\mathbb{G}_T}$

bilinear: $e([a]P,[b]Q) = e(P,[b]Q)^a = e([a]P,Q)^b = e(P,Q)^{ab}$



[1] Courtesy of D. Aranha for the tikz figure.

Pairings out-circuit

ate pairing

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$$
$$(P, Q) \mapsto f_{t-1,Q}(P)^{(q^k-1)/r}$$

- $f_{t-1,Q}(P)$ is the Miller function
- $f \mapsto f^{(q^k-1)/r}$ is the final exponentiation

Examples: For polynomial families in the seed x,

BLS12 $e(P, Q) = f_{x,Q}(P)^{(q^{12}-1)/r}$

BLS24 $e(P, Q) = f_{x,Q}(P)^{(q^{24}-1)/r}$

Pairings out-circuit: Miller algorithm

Definition

Miller algorithm computes the Miller function $f_{s,Q}$ such that Q is a zero of order s and $[s]Q$ is a pole of order 1, i.e.

$$div(f_{s,Q}) = s(Q) - ([s]Q) - (s-1)\mathcal{O}$$

For integers i and j,

$$f_{i+j,Q} = f_{i,Q} f_{j,Q} \frac{\ell_{[i]Q,[j]Q}}{v_{[i+j]Q}},$$

where $\ell_{[i]Q,[j]Q}$ and $v_{[i+j]Q}$ are the two lines needed to compute $[i+j]Q$ from $[i]Q$ and $[j]Q$ (ℓ intersecting the two points and v the vertical).

Pairings out-circuit: Miller algorithm

Algorithm 1: MillerLoop(s, P, Q)
Output: $m = f_{s,Q}(P)$
$m \leftarrow 1$; $R \leftarrow Q$
for *b from the second most significant bit of s to the least* **do**
  $\ell \leftarrow \ell_{R,R}(P)$; $R \leftarrow [2]R$; $v \leftarrow v_{[2]R}(P)$ Doubling Step
  $m \leftarrow m^2 \cdot \ell / v$
  if $b = 1$ **then**
    $\ell \leftarrow \ell_{R,Q}(P)$; $R \leftarrow R + Q$; $v \leftarrow v_{R+Q}(P)$ Addition Step
    $m \leftarrow m \cdot \ell / v$
return m

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  $m \leftarrow m^2 \cdot \ell$

  if $b = 1$ **then**

    $\ell \leftarrow \ell_{R,Q}(P)$; $R \leftarrow R + Q$;   Addition Step

    $m \leftarrow m \cdot \ell$

return m

Pairings out-circuit: Miller algorithm

$\mathbb{G}_2$: - Coordinates compressed in $\mathbb{F}_{q^{k/d}}$ instead of $\mathbb{F}_{q^k}$ (where d is the twist degree) [BN06]
- Homogeneous projective coordinates (X, Y, Z) [AKL+11, ABLR14]
- Sharing computation between Double/Add and lines evaluation [AKL+11, ABLR14]

Finite fields: - $\mathbb{F}_p \to \cdots \to \mathbb{F}_{p^{k/d}} \to \cdots \to \mathbb{F}_{p^k}$
- efficient representation of line (multiplying the line evaluation by a factor $\to$ wiped out later) [ABLR14]
- efficient sparse multiplications in $\mathbb{F}_{p^k}$ [Sco]

Pairings out-circuit: Final exponentiation

$$\frac{p^k-1}{r} = \underbrace{\frac{p^k-1}{\Phi_k(p)}}_{\text{easy part}} \cdot \underbrace{\frac{\Phi_k(p)}{r}}_{\text{hard part}}$$

easy part: a polynomial in p with small coefficients (Frobenius maps)
e.g. (BLS12): `1F2 + 1Conj + 1Inv + 1Mul` in $\mathbb{F}_{p^{12}}$

hard part: More expensive. Vectorial or lattice-based
Optimizations [HHT, AFK$^+$13, GF16]
dominating cost: `CycloSqr [GS10, Kar13] + Mul` in $\mathbb{F}_{p^k}$

Overview

Example: Pairing on inner BLS12

$(E/\mathbb{F}_{p(x)}) : Y^2 = X^3 + 1$

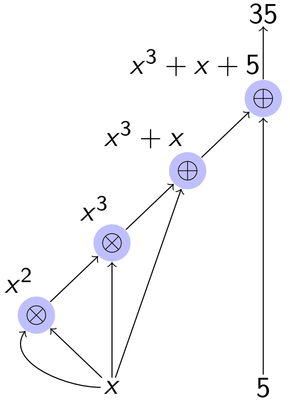
$$r(x) = x^4 - x^2 + 1; \quad p(x) = (x-1)^2 \cdot r(x)/3 + x; \quad t(x) = x + 1$$

with $x \equiv 1 \mod 3 \cdot 2^L$ (input $L \in \mathbb{N}^*$ the desired 2-adicity).
e.g. for BLS12-377 $x =$ 0x8508c00000000001.

- $k = 12 \implies \mathbb{G}_T$ over $\mathbb{F}_{p^{12}}$
- $d = 6 \implies \mathbb{G}_2$ over $\mathbb{F}_{p^2} = \mathbb{F}_{p^{k/d}}$
- $\mathbb{F}_p \to \mathbb{F}_{p^2} \to \mathbb{F}_{p^6} \to \mathbb{F}_{p^{12}}$ or $\mathbb{F}_p \to \mathbb{F}_{p^2} \to \mathbb{F}_{p^4} \to \mathbb{F}_{p^{12}}$
  (better compression ratio: 1/3 with XTR or CEILIDH)
- $e(P, Q) = f_{x,Q}(P)^{(p(x)^{12}-1)/r(x)}$ (best optims. [ABLR14, HHT, GS10])

Rank-1 Constraint System

$x^3 + x + 5 = 35 \qquad (x = 3)$



constraints:

$$o = l \cdot r$$

$$a = x \cdot x$$
$$b = a \cdot x$$
$$c = (b + x) \cdot 1$$
$$d = (c + 5) \cdot 1$$

witness:

$$\vec{w} = \begin{pmatrix} \text{one} & x & d & a & b & c \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 3 & 35 & 9 & 27 & 30 \end{pmatrix}$$

Algorithm optimizations

	Time	Constraints
BLS12-377	< 1 ms	$\approx$ **80 000**

- Miller loop:
  - Affine coordinates $\rightarrow \approx$ 19k (`arkworks`)
  - Division in extension fields
  - Double-and-Add in affine
  - lines evaluations (1/y, x/y)
  - Loop with short addition chains
  - Torus-based arithmetic
- Final Exponentiation:
  - Cyclotomic squarings
  - Torus-based arithmetic
  - Exp. with short addition chains

19k $\rightarrow \approx$ 11k (`gnark`)

R1CS is about writing $o = l \cdot r$

- Over $\mathbb{F}_p$ ($\mathbb{F}_r$ of BW6):
  - Square = Mul ($o = l \cdot l$)
  - Inv = Mul + 1C ($1/l = o \rightarrow 1 \stackrel{?}{=} l \cdot o$ with o an input hint)
  - Div = Mul + 1C ($r/l = o \rightarrow r \stackrel{?}{=} l \cdot o$ with o an input hint)
  - Inv+Mul $\rightarrow$ Div
- Over $\mathbb{F}_{p^e}$:
  - Square $\neq$ Mul (e.g. $\mathbb{F}_{p^2}$ 2C vs 3C)
  - Inv = Mul + eC ($1/l = o \rightarrow 1 \stackrel{?}{=} l \cdot o$ with o an input hint)
  - Div = Mul + eC ($r/l = o \rightarrow r \stackrel{?}{=} l \cdot o$ with o an input hint)
  - Inv+Mul $\rightarrow$ Div

Optimizations

Affine arithmetic

$\mathbb{G}_2$ Double: $[2](x_1, y_1) = (x_3, y_3)$

$$\lambda = 3x_1^2/2y_1$$
$$x_3 = \lambda^2 - 2x_1$$
$$y_3 = \lambda(x_1 - x_3) - y_1$$

$\mathbb{G}_2$ Add: $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$

$$\lambda = (y_1 - y_2)/(x_1 - x_2)$$
$$x_3 = \lambda^2 - x_1 - x_2$$
$$y_3 = \lambda(x_2 - x_3) - y_2$$

	Div (5C)	Sq (2C)	Mul (3C)	total
Double	1	2	1	12C
Add	1	1	1	10C

Tailored optimization: Short addition chain of the seed x with inverted Double/Add wieghts! (cf. `github.com/mmcloughlin/addchain`)

In the Miller loop, when $b = 1 \implies [2]R + Q \to$ **22C**
Instead: $[2]R + Q = (R + Q) + R \to$ **20C**
Better: omit y_{R+Q} computation in $(R + Q) + R \to$ **17C** [ELM03]
$\mathbb{G}_2$ Double-and-Add: $[2](x_1, y_1) + (x_2, y_2) = (x_4, y_4)$

$$\lambda_1 = (y_1 - y_2)/(x_1 - x_2)$$
$$x_3 = \lambda_1^2 - x_1 - x_2$$
$$\lambda_2 = -\lambda_1 - 2y_1/(x_3 - x_1)$$
$$x_4 = \lambda_2^2 - x_1 - x_3$$
$$y_4 = \lambda_2(x_1 - x_4) - y_1$$

	Div (5C)	Sq (2C)	Mul (3C)	total
Double-and-Add	2	2	1	**17C**

Optimizations

lines evaluation

- ℓ is $ay + bx + c = 0 \in \mathbb{F}_{p^2}$
- $\ell_{\psi([2]R)}(P)$ and $\ell_{\psi(R+Q)}(P)$ are of the form $(a'y_P, 0, 0, b'x_P, c', 0) \in \mathbb{F}_{p^{12}}$ $(\psi : E'(\mathbb{F}_{p^{k/d}}) \to E(\mathbb{F}_{p^k}))$ [ABLR14]
  $\to$ sparse multiplication (1) in $\mathbb{F}_{p^{12}}$
- precompute $1/y_P$ (5C) and x_P/y_P (5C) and $\ell(P)$ becomes $(1, 0, 0, b'x_P/y_P, c'/y_p, 0) \in \mathbb{F}_{p^{12}}$
  $\to$ better sparse multiplication (2) in $\mathbb{F}_{p^{12}}$

	total
Full Mul	54C
Sparse Mul (1)	39C
Sparse Mul (2)	30C

Easy part:

```
 t.Conjugate(m)
m.Inverse(m)  // 66C
 t.Mul(t, m)   // 54C
m.FrobeniusSquare(t)
m.Mul(m, t)   // 54C
```

Easy part:

```
t.Conjugate(m)
t.Div(t, m) // 66C
m.FrobeniusSquare(t)
m.Mul(m, t) // 54C
```

Easy part: (more on that later)

```
t.Div(-m[0], m[1]) // 18C
m.TorusFrobeniusSquare(t)
m.TorusMul(m, t) // 42C
r := Decompress(m) // 48C
```

	total
Old	174
New	120
New (Torus)	60 (or 108)

Hard part (Hayashida et al. [HHT])

```
t[0].CyclotomicSquare(m)
t[1].Expt(m) // m^x addchain (Mul + CycloSqr)
t[2].Conjugate(m)
t[1].Mul(t[1], t[2])
t[2].Expt(t[1])
t[1].Conjugate(t[1])
t[1].Mul(t[1], t[2])
t[2].Expt(t[1])
t[1].Frobenius(t[1])
t[1].Mul(t[1], t[2])
m.Mul(m, t[0])
t[0].Expt(t[1])
t[2].Expt(t[0])
t[0].FrobeniusSquare(t[1])
t[1].Conjugate(t[1])
t[1].Mul(t[1], t[2])
t[1].Mul(t[1], t[0])
m.Mul(m, t[1])
```

Table: Square in cyclotomic $\mathbb{F}_{p^{12}}$

	Compress	Square	Decompress
Normal	0	36	0
Granger-Scott [GS10]	0	18	0
Karabina [Kar13] SQR2345	0	12	19
Karabina [Kar13] SQR12345	0	15	8
Torus ($\mathbb{T}_2$)[RS03]	24	24	48

- 1 or 2 squarings $\implies$ Granger-Scott
- 3 squarings $\implies$ Karabina SQR12345
- $\geq$ 4 squarings $\implies$ Karabina SQR2345

Table: Mul in cyclotomic $\mathbb{F}_{p^{12}}$

	Compress	Multiply	Decompress
Normal	0	54	0
Torus ($\mathbb{T}_2$)[RS03]	24	42	48

- Compression/Decompression only once!
- Whole final exp. in compressed form over $\mathbb{F}_{p^6}$
- Better:
  - Absorb the compression in the easy part computation
  - Do we really need decompression?

Definition

Let $\mathbb{F}_q$ be a finite field and $\mathbb{F}_{q^k}$ a field extension of $\mathbb{F}_q$. Then the norm of an element $\alpha \in \mathbb{F}_{q^k}$ with respect to $\mathbb{F}_q$ is defined as the product of all conjugates of α over $\mathbb{F}_q$, namely $N_{\mathbb{F}_{q^k}/\mathbb{F}_q} = \alpha\alpha^q \cdots \alpha^{q^{k-1}} = \alpha^{(q^k-1)/(q-1)}$

$$T_k(\mathbb{F}_q) = \bigcap_{\mathbb{F}_q \subset F \subset \mathbb{F}_{q^k}} ker(N_{\mathbb{F}_{q^k}/F})$$

Lemma

Let $\alpha \in \mathbb{F}_{q^k}$, then $\alpha^{(q^k-1)/\Phi_k(q)} \in T_k(\mathbb{F}_q)$

$\mathbb{T}_2$ cryptosystem introduced by Rubin and Silverberg [RS03].
Let $\alpha = c_0 + \omega c_1 \in \mathbb{F}_{q^k} - \{1, -1\}$ (cyclotomic subgroup), we have

compress	$f(\alpha) = (1 + c_0)/c_1 = \beta \in \mathbb{F}_{q^{k/2}}$
decompress	$f^{-1}(\beta) = (\beta + \omega)/(\beta - \omega) = \alpha$
Mul	$\beta_1 \times \beta_2 = (\beta_1\beta_2 + \omega)/(\beta_1 + \beta_2)$
Square	$\beta^2 = \frac{1}{2}(\beta + \omega/\beta)$
Inverse	$1/\beta = -\beta$

$\mathbb{T}_2$ **arithmetic is R1CS-friendly!**

Easy part: $m^{(q^{12}-1)/\Phi_k(p)} = m^{(p^6-1)(p^2+1)}$
Let $\alpha = c_0 + \omega c_1 \in \mathbb{F}_{q^{12}} - \{1\}$ (cyclotomic subgroup),

$$\begin{aligned}
\alpha^{p^6-1} &= (c_0 + \omega c_1)^{p^6-1} \\
&= (c_0 + \omega c_1)^{p^6} / (c_0 + \omega c_1) \\
&= (c_0 - \omega c_1)/(c_0 + \omega c_1) \\
&= (-c_0/c_1 + \omega)/(-c_0/c_1 - \omega) \\
f(\alpha) &= (-c_0/c_1)^{p^2+1} \\
&= (-c_0/c_1)^{p^2} \times (-c_0/c_1)
\end{aligned}$$

$\rightarrow$ 60C

Carry the whole Miller loop in compressed form (e.g. [NBS08])

- Isolate $m = 1$ (just $m = \ell \rightarrow$ less constraints)
- Write m as: $f(m) = (-c_0/c_1)^{p^2} \times (-c_0/c_1)$
- Use $\mathbb{T}_2$ cyclotomic squaring
- Write lines as

$$(1, 0, 0, b'x/y, c'/y, 0) \in \mathbb{F}_{p^{12}} \mapsto -1/(b'x/y + \omega c'/y)^{p^2+1} = -1/D \in \mathbb{F}_{p^6}$$

- Cyclotomic sparse Mul as:

$$f(m) \times f(\ell) = (f(m)f(\ell) + \omega)/(f(m) + f(\ell))$$
$$= (-f(m) + \omega D)/(f(m)D + 1)$$

Conclusion

Implementation open-sourced (MIT/Apache-2.0) at
https://github.com/ConsenSys/gnark
e.g. For BLS12-377,

	Constraints
Pairing	**11535**
Groth16 verifier	19378
BLS sig. verifier	14888
KZG verifier	20679

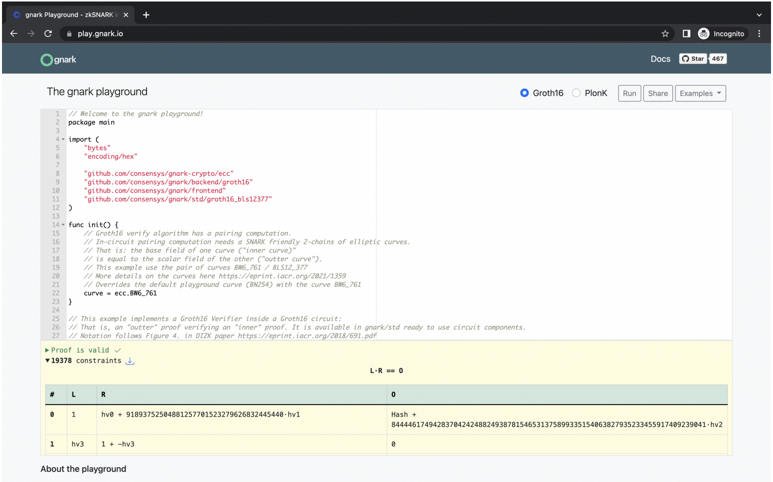
For BLS24-315, a pairing is **27608** contraints .
More optimizations in mind:

- Quadruple-and-Add Miller loop [CBGW10]
- Fixed argument Miller loop (KZG, BLS sig) [CS10]
- ~~Longa's sums of products Mul [Lon22]~~

Conclusion

Let's play with gnark!

https://play.gnark.io/



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